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Double-Blind Tests Program for Astrometric Planet Detection with GAIA.

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Summary

This document describes the test program devised to assess GAIA's capability in detecting and measuring exoplanets. It implements the recommendations of the members of the Planetary System Working Group (PSWG) that met in Paris in April 2002 and in Gressoney (Sep 2002). This test program will be promptly started after consensus is reached among the participating groups on the procedure itself.

Section 1 provides the basic definitions and sets the goals of the study. Ultimately, we expect not only to learn what GAIA can achieve, but also how to best deal with real data when they become available, and even to integrate our analysis tools within the prototype GAIA data reduction system (GDAAS). Section 2 introduces the idea of the two Work Phases foreseen: "Phase 0", mainly a software readiness test, is detailed in Section 3, while Section 4 presents "Phase 1", the study phase which will implement the double-blind test strategy to probe GAIA's capability with exoplanets. Sec. 5 lists the participating groups and describes who does what and where. Finally, Section 6 discusses the simulation assumptions and the simulated data (and their organization), which will be delivered to the "Solver" teams and to the "Evaluator".

For the benefit of all the participants and to facilitate a swift understanding of what the simulation does, its mathematical model is provided in the Appendix.

1 Definition and Goals

Our recent exploratory work (e.g., Lattanzi et al. 2000, Sozzetti et al 2001, and 2002) has shown in some detail what high-accuracy astrometric missions, such as GAIA and SIM, can achieve in terms of search, detection and measurement of extrasolar planets of mass ranging from Jupiter-like to Earth-like. In these studies we have adopted a qualitatively correct description of the measurements that each mission will carry out, and we estimated detection probabilities and orbital parameters using realistic, non-linear least squares fits to those measurements. For GAIA, we used the then-current scanning law and error model; for SIM, we included reference stars as well as the target, and included realistic observational overheads and signal-to-noise estimates as provided by the SIM Project.

Although valid and useful, the studies currently available need updating and improvements. Thus far, we have largely neglected the difficult problem of selecting adequate starting values for the non-linear fits, using perturbed starting values instead. The study of multiple-planet systems, and in particular the determination of whether the planets are coplanar—within suitable tolerances—is incomplete. The characteristics of GAIA have changed, in some ways substantially, since Sozzetti et al (2001). Last but not least, simulations and analysis have been carried out thus far by a single team—ours—thus raising the issue of blind tests, for which simulations and analysis are truly independent.

We present here a substantial program of double-blind tests, with the twin goals of extending and updating the existing results and of establishing a distributed network of groups capable of producing simulations and analysis in a true unbiased, independent setup. We also plan to upgrade the simulation and analysis of observations by including an increasingly realistic description of the measurement process, with additional instrumental effects and with the calibration imperfections that can be expected for GAIA. Ultimately, we expect not only to learn what can be achieved with GAIA, but also how to best study real data when they become available, and even to integrate our analysis tools within the prototype network of GAIA data reduction processes (GDAAS).

The tests will be conducted in double-blind mode, with three groups of participants:

1. *Simulators*: the group(s) that define and generate the simulated observations, using clearly stated assumptions on the observation process; simulators also define the type of results that are expected for each set of simulations.

2. *Solvers*: the group(s) that receive the simulated data and produce "solutions"—as defined by the simulators; solvers define the criteria they adopt in answering the questions posed by the simulators.

3. *Evaluators*: the group(s) that receive both the "truth"—i.e., the input parameters—from the simulators and the solutions from the solvers, compare the two, and draw a first set of conclusions on the process.

We envision a sequence of tasks, each with well-defined goals and time scales. Each task requires a separate set of simulations, and is carried out in several steps:

1. Simulation: The Simulators make the required set of simulations available to the Solvers, together with a clear definition of the required solutions.

2. Clarification: A short period (typically one week) after the simulations are made available in which the Solvers request any necessary clarification on the simulations themselves and on the required solutions; after the clarification period, there should be no contact between Simulators and Solvers until the Discussion step.

3. Delivery: On a specified deadline, the Simulators deliver the input parameters for the simulations to the Evaluator, and the Solvers deliver their solutions together with a clear explanation of the criteria they used—e.g., the statistical meaning of "detection", or how parameter uncertainties were defined.

4. Evaluation: The Evaluators compare input parameters and solutions and carry out any statistical tests they find useful, both to establish the quality of the solutions and to interpret their results in terms of the capabilities of GAIA, if applicable.

5. Discussion: The Evaluators publicize their initial results. All participants are given access to input parameters and all solutions, and the Evaluators' results are discussed and modified as needed.

6. Post-mortem: All participants discuss what did and did not work in their approach, and lessons learned are compared. If substantial changes are needed either in software or in procedures, the participants agree on a waiting period until the start of the next task.

7. Report: A detailed report is prepared collegially, with inputs from all parties, and edited by the WG coordinators.

We expect a typical task to take roughly two to three months of actual work, with a short coda to prepare the report—the preparation of the report will typically overlap with the work on the following task, and in some cases results from more than one task may be combined into a single report. Of course, we expect that our somewhat naïve initial description will evolve as the work progresses and we learn more on what procedures work best for our relatively small working group.

2 Phases of the work

The test program we propose will initially adopt a simplified model for the process of observing stellar positions with GAIA. Much is yet to be defined in terms of the GAIA error model, observation parameters, and data analysis, and thus we prefer to start with a purely geometric model in which the error in each individual measurement is described by a single number, without known correlations with other errors or quantities. Somewhat simplistically, we assume that each time a star is observed with GAIA its position ψ along the instantaneous great circle IGC is measured with a Gaussian error distribution of known dispersion σ_ψ, while the parameters of the IGC itself are known without error and no measurement is made on the position η of the star perpendicular to the IGC. We also assume that external elements that affect the measurement, such as aberration, general-relativistic effects (the deflection parameter γ, gravitational lensing), and uncertainties in the local reference system and in the sphere solution are eliminated by the standard data analysis process. In the first phase of the work, we also neglect perspective acceleration—equivalently, we assume that all stars have zero heliocentric radial velocity. The measurement process under these assumptions is described in detail in the Appendix.

In a second phase of the test program, these assumptions will be progressively relaxed and more realistic error models, including zero point uncertainties, error propagation from the IGC solution, and a realistic error distribution for ψ including possible bias and magnitude terms. This activity will necessarily be tied with further developments on the understanding of the technical specifications of GAIA and its instruments, and of its observation and data analysis process; therefore we refrain from further specifications on its details at this time.

Prior to the start of Phase 1 of the test program, we envision a readiness test, which we call Phase 0, to ensure that all procedural aspects of the test are resolved—e.g., file formats are defined, the interpretation of simulation data is clear, the specifications of the geometric model of the measurement process are agreed upon, and so on—and that the software tools are all compatible and ready to go. This initial Phase 0 is expected to take about two months.

3 Phase 0: Goals, tasks, and timelines

As stated, Phase 0 has the main goal to ensure that all the machinery of the test process is in place and ready to operate, and that the simulations and solution tools are compatible. The tests will be kept small in size—a few hundred simulations for each—so that there will be time to study and understand any anomalies and disagreements. Phase 0 will consist of two tasks: verification of planet detection on the basis of the χ^2 test, and verification of the orbital fit with given starting parameters. Both tests reflect results that are already available in the literature, and therefore do not probe significant new ground.

3.1 Task T0a: Verify the single-star fit and the χ^2 detection test

Task T0a will consist of 1000 simulations, where we define one simulation as the time sequence of the observations for one stellar target, comprising an average of 42 observation epochs through the 5-year mission duration. About 50% of the stars will have a single planet, with a signature significance (α/σ) of 5 to 10; orbital parameters of the planet will be randomized, with period ranging from ~ 1 month to comparable to the mission duration. Stellar parameters will likewise be randomized. Solvers will be asked to carry out the single-star fit for each star, and report the stellar parameters they derive as well as the χ^2 of the solution; they will also be asked which stars fail the χ^2 test and with what significance.

3.2 Task T0b: Verify star+planet fit

Task T0b will consist of 1000 simulations of stars with a single planet each. Planets will have high signature significance, $\alpha/\sigma \sim 100$, and the same period range as in Task 0b. The planet parameters will be made available as starting values for the star+planet solution, possibly with some noise added. Solvers will be asked to produce the best-fitting star+planet solution, with the corresponding model values of ψ, and the quality of the solution based on the χ^2 test.

3.3 Success criteria and tentative schedule

Success of the Phase 0 tests entails correction of any procedural glitches that may arise, and agreement between the solutions and results of different teams to the degree expected after accounting for differences in the solution process. When both tests are completed successfully, the teams will be ready to continue on to Phase 1 of the test program, which will produce new and relevant results.

A possible schedule for Phase 0 is:

- Day 0: Simulations for both Tasks are made available online. Solvers notify Simulators of any problems within 7 days.
- Day 7: Clarification period ends; no more contacts between Simulators and Solvers until Day 56.
- Day 21: Solvers send their solutions for Task T0a to Evaluators. Simulators send input parameters for Task T0a to Evaluators.
- Day 42: Solvers send their solutions for Task T0b to Evaluators. Simulators send input parameters for Task T0b to Evaluators.
- Day 56: Evaluators circulate initial analysis of solutions, and assess degree of success. Simulators make input parameters available to all. Solvers make solutions available to all. Discussion starts.
- Day 63: Evaluation finalized. Success or failure declared. Post-mortem carried out (probably via teleconference). Report started.

4 Phase 1: Goals, tasks and schedules

Phase 1 of our program encompasses those tests that are necessary to establish a realistic estimate of the search and measurement capabilities of GAIA, while maintaining the simplified, geometric model of the measurement process described in the Appendix. At the end of Phase 1, the participating groups will be able to analyze data produced by a nominal satellite, without taking into account the imperfections due to measurement biases, non-Gaussian error distributions, imperfections in the sphere solution, and so on. They will also be in a position to convert any Gaussian error model for GAIA measurements into expected detection probability—including completeness and false positives—and accuracy in orbital parameters that can be achieved within the mission. They will be able to assess to what extent, and with what reliability, coplanarity of multiple planets can be determined, and how the presence of a planet can degrade the orbital solution for another. In other words, the participating groups will be able to determine GAIA's ability to detect and measure planets under realistic analysis procedures, albeit in the presence of an idealized measurement process.

We break down the work plan for Phase 1 into four tasks: T1, T1b, T2, and T3. Task T1 will probably be the most challenging in terms of mass of data to analyze and processing time; task T3 will probably be the most complex. We expect both T1 and T3 to require 3 months each. Tasks T1b and T2 will probably require 2 months each.

4.1 Task T1: Planet detection and its significance

Task T1 will consist of the analysis of a massive number of simulated observations, probably about 10^5 stars, in order to establish under what conditions the presence of a planet can be detected, and with what reliability. The simulations will consist of a mix of stars with no or one planet, in roughly equal numbers, possibly with a small number of multiple-planet cases. Signature significance will range from 0.1 to 10, thus going from the non-detectable to the "easily" detectable. Periods short, medium, and long will all be represented, with short periods essentially unresolved by the mission, medium periods well-resolved and well-sampled, and long periods sampled only for a fraction of their orbital motion. Each "bin" of period and significance should be occupied by several hundred cases, each with orbital parameters distributed randomly over the relevant range, so that detection probability can be established with confidence.

Each Solver group will be free to establish their own detection test, with a significance level of their choice. If desired, each Solver group can establish more than one detection test, as long as stars are clearly identified as single or non-single under each and every test independently.

4.2 Task T1b: do orbital fits improve planet detection?

Many detection tests, such as those based on the quality of the single-star fit, establish a balance between detection threshold and probability of false positives. For example, raising the threshold of detection on a simple χ^2 test from 95% significance to 99% significance will reduce the number of false positives expected by a factor of 5, but at the cost of raising the level of signature significance for which completeness can be achieved—in other words, decreasing the sensitivity. The "optimal" threshold for the detection test depends on the goals of the project, the frequency of detectable planets, and other considerations, but undoubtedly a significant number of false positives if undesirable.

It has been suggested that an additional confirmation for marginally detected planets, which could just as well be false positives, can be achieved by fitting a star+planet model to the observations. If the planet is real, the star+planet fit should perform significantly better in a measurable way; if the planet is a statistical fluke, the improvement should be less pronounced. The star+planet fit could thus act as an additional filter to weed out false positives without substantially worsening the sensitivity of the test.

Task T1b would establish whether such a filter is in fact useful. Each Solver group would choose "borderline" detections among the cases studied for Task T1, and proceed with a star+planet fit for each case. Solver groups would then decide on an additional criterion (Goodness–of–fit, likelihood ratio, etc) to identify detections. Of course, the input parameters for Task T1 would not be communicated until the results of T1b are delivered. The Evaluator will then compare the detection results of T1 and T1b to determine whether an orbital fit is a useful coadiuvant of a plain detection test.

4.3 Task T2: Parameters of star+planet orbital fit

Task T2 will determine how well the orbital parameters of a single planet can be measured for a variety of signature significance, period, inclination, and other parameters. The simulations will consist of $\sim 10^4$ stars, each with a single planet with significance ranging from 3 (barely detected) to several hundred. Solvers will determine the best-fit orbital parameters, together with an error estimate for each and covariances if appropriate. Evaluators will first assess the quality of the solutions and of their error estimates. Evaluators will then study the distribution of orbital parameter errors vs. the stellar and orbital parameters themselves, with the goal of deriving simple expressions that can predict the accuracy of the orbital solution for various types of planets as a function of the GAIA error model.

4.4 Task T3: Multiple planet solution and coplanarity

Task T3 will determine how well multiple planets can be identified and solved for, as well as how well their coplanarity can be established. In addition, the accuracy of multiple-planet solutions will be compared with that of single-planet solutions for planets with comparable properties. The Task will be based on $\sim 10^3$ simulations of stars with two to four planets each, including a small number of stars with a single planet. Planets will be assumed to be strictly non-interacting, in that each planet follows pure Keplerian orbits around the center-of-mass of the system. Solvers will determine how many planets can be detected and solved for in each case, as well as whether they are coplanar within a prespecified tolerance. The method to be used for the coplanarity test is up to the Solvers; one possibility is a Likelihood Ratio test between a coplanar and a general solution. Solvers will also produce error estimates, and covariances if appropriate, for the orbital parameters they determine. Simulators will ensure that some simulations have a dominant planet with parameters similar to those of a case studied in Test T2, so that the quality of the single- and multiple-planet solution can be compared on a case-by-case basis. (Alternatively, single-planet simulations and solutions for the dominant planet in each case can be carried out after the fact.) Evaluators will 1) assess the quality of the solutions and of the estimated errors; 2) study the distribution of orbital error parameters in comparison with the single-planet case, and 3) assess the quality and reliability of the coplanarity test.

4.5 Assessment of GAIA impact on exoplanet research; additional tasks and interpretation

After completing the above Tasks, the participants will be in an ideal position to assess the scientific and technical capabilities of GAIA globally in the light of the dominant open questions in exoplanet research. We will identify the areas in which GAIA can be expected, on the basis of our results, to have a dominant impact, and delineate a small number of recommended research programs that can be conducted successfully by the mission as planned. We will also identify the mission parameters (e.g., the elementary measurement error, the scanning law parameters) that most clearly affect the capabilities of GAIA for exoplanet research, and define the floor below which the impact of GAIA will be clearly impaired, as well as areas in which modest improvements can greatly improve the results expected from GAIA. If appropriate, we will carry out small additional tasks targeted towards a better understanding of the recommended research programs.

4.6 Results, success metric and tentative schedule

Ideally, the results expected from Phase 1 of the project include:

1. An improved, more realistic assessment of the detectability and measurability of single and multiple planets under a variety of conditions, parametrized by the sensitivity of GAIA;

2. One or more reports describing these results in detail, thus answering the mandate of the Planet Working Group;

3. Possibly the preparation of one or more publications for refereed journals, authored by all participating parties, which will make the results of the study available to the whole astronomical community;

4. An assessment of the impact of GAIA in critical areas of planet research, in dependence on its expected capabilities;

5. The establishment of several Centers with a high level of readiness for the analysis of GAIA observations relevant to the study of exoplanets

Success will be achieved if a substantial fraction of the above items, especially 1 and 2, are completed in collaboration by several Centers.

The schedule of Phase 1 is of necessity less predictable than that of Phase 0. Based on the complexity and size of each task, we estimate that Task T1 will be completed about 3 months after the start of Phase 1; Task T1b 4.5 months after the start; Task T2 7 months after the start; and Task T3 about 10 months after the start. The assessment of the capabilities of GAIA will take place over about two months, together with any small additional tasks that may be required for a further refinement of our results. Overall, Phase 1 of the project will last approximately one year. This schedule assumes no delays due to necessary software and procedure upgrades; a more conservative schedule would allow for 3–6 months for such upgrades, resulting in a total estimated length of 15–18 months.

5 Current status of the project: who, what, where

Several researchers have already made explicit committments to carry out a substantive research and test program over the next two years. At present, the likely assignments are as follows:

- Torino 1 (Lattanzi, Spagna): Definition and oversight of simulations

- Torino 2 (Morbidelli, Pannunzio): Generation of simulations; data management, archiving, and distribution

- Cambridge, USA (Sozzetti): Solver
- Helsinki 1 (K. Muinonen, P. Muinonen): Solver
- Heidelberg (Bernstein): Solver
- Brussels (Pourbaix): Solver
- Helsinki 2 (M. Kaasalainen): Solver
- Baltimore (Casertano): Evaluator

The revised simulation procedures to implement this experiment are ready at the Observatory of Torino (OAT). Most groups have already developed solution procedures, but some improvements and adaptations may be necessary. Evaluation procedures descend from those developed for the Lattanzi et al. and Sozzetti et al. papers, but will need refining as well. Agreements on formats and procedures need to be achieved, but no substantial difficulties exist. Phase 0 will start on April 22 and Phase 1 on June 2,if Phase 0 tests are successful.

6 Simulation Setup

6.1 Assumptions and Experiment Definition

Main *a priori* assumptions are:

- the position of the pole of each IGC is considered known a priori (perfect attitude);
- the IGC abscissa ψ is only affected by random errors; no systematic effects are considered (e.g., zero-point errors, chromaticity, etc...);
- light aberration, light deflection, and other apparent effects are as if they were perfectly removed from the observed abscissa

The scanning law for the time being is exactly the one devised for GAIA, i.e. precession angle around the Sun direction $\xi = 50°$, precession speed of the satellite's spin axis $V = 5.22$ rev/yr, spin axis rotation speed 60 arcsec/sec. We assume in our double-blind tests program that detectors behave in a way that astrometric errors still scale with magnitude at V $\sim$ 12, and adopt a single-measurement error σ_ψ defined by:

$$\sigma_\psi = \frac{\sigma_{\rm fin} * \sqrt{N_{\rm obs}}}{f_g} \tag{1}$$

If the end-of-mission error $\sigma_{\rm fin}$ is 11 μas at V = 15 (as per Astrium tables), the geometrical factor $f_g = 2.2$, and $N_{\rm obs} = 42$, and assuming a scaling factor of 0.25 for V $\sim$ 12, then the constant single-measurement error $\sigma_\psi \simeq 8$ μas. This value reflects the changes in the present scanning law with respect to the one envisaged before (less observations per object, but longer integration times, for a globally unchanged total observing time spent on each given target).

The mission lifetime is set to 5 years, and the ecliptic longitude of the Sun (with the Earth assumed to go about the Sun in a perfectly round orbit) is $\lambda_\odot = 90°$ at the catalogue reference epoch $t_0 = 2.5$ years.

The values of the astrometric parameters are drawn from simple distributions, not resembling any specific galaxy model. The distribution of ecliptic coordinates is random, uniform. The distribution of proper motions is gaussian, with dispersion equal to a value of transverse velocity $V_T = 15$ km/sec, typical of the solar neighborhood.

As for what concerns the other relevant parameters, we will produce experiments where the stellar mass is for simplicity always kept fixed to 1 $M_\odot$, and express detection probabilities and the efficiency in orbit reconstruction as a function of distance, orbital elements, planet mass, and astrometric signal-to-noise ratio α/σ_ψ, where $\alpha = (M_p \times a_p)/(M_s \times D)$ is the astrometric signature (M_p and M_s are the planet and stellar mass, a_p is the planet's orbital semi-major axis, and D is the distance). Clearly, the results will also be a function of the details of the fitting procedures adopted by the different teams, which is one of the questions we are addressing with this testing program.

6.2 Simulation Output

As a result of the definite boundaries given to the experiments to perform, the main output from the simulation of GAIA observations will be two files, the first (the "Observations file", with extension "IGC") containing, for each observation (record) the following quantities:

(1) $N_\star$ = the identifier of the observed target

(2) N_{IGC} = the identifer of the Instantaneous Great Circle (IGC) on which the given star was observed

(3) λ_p = longitude of the pole of the IGC (rad)

(4) β_p = latitude of the pole of the IGC (rad)

(5) t = the observation epoch (the same for all stars on the same IGC) (year)

(6) $\psi_{\rm obs}$ = the observed abscissa (rad)

(7) $\sigma_{\psi_{\rm obs}}$ = rms error of the observed abscissa (rad)

(8) $\lambda_\odot$ = the longitude of the mean Sun (rad)

and the second one (the "Catalog file", with extension "CATL") containing, for each target star (record), the parameters:

(a) $N_\star$ = the id of the observed target

(b) λ_b = ecliptic longitude of system barycenter (rad)

(c) β_b = ecliptic latitude of system barycenter (rad)

(d) μ_λ = long. component of proper motion (rad/yr)

(e) μ_β = lat. component of proper motion (rad/yr)

(f) π = parallax (rad)

(g) t_0 = catalogue reference epoch (yr)

The five astrometric parameters will be provided as nominal/catalog values, with uncertainties typical of the **Hipparcos Catalogue** *at some appropriate epoch.*

APPENDIX

The simulated model

The code for the simulation of GAIA observations is (for the time being) run by the Torino Observatory.

We start by generating spheres of N targets. Each target's two-dimensional position is described in the ecliptic reference frame via a set of two coordinates λ_b and β_b, called here barycentric coordinates. We linearly update the barycentric position (of the star + planet system) as a function of time, accounting for the (secular) effects of proper motion (two components, μ_λ and μ_β), the (periodic) effect of the parallax π, and the (Keplerian) gravitational perturbations induced on the parent star by one or more orbiting planets (mutual interactions between planets are presently not taken into account). The model of motion can thus be expressed as follows:

$$\mathbf{x}_{\rm ecl} = \mathbf{x}^0_{\rm ecl} + \mathbf{x}^{\pi,\mu}_{\rm ecl} + \sum_{j=1}^{n_p} \mathbf{x}^{\rm K,j}_{\rm ecl} \tag{2}$$

Where:

$$\mathbf{x}^0_{\rm ecl} = \begin{pmatrix} \cos\beta_b \cos\lambda_b \\ \cos\beta_b \sin\lambda_b \\ \sin\beta_b \end{pmatrix}$$

is the initial position vector of the system barycenter. The various perturbative effects are initially defined in the tangent plane. The parallax and proper motion terms are contribute as:

$$\mathbf{x}_{\pi,\mu} = \begin{pmatrix} \mu_\lambda t + \pi F_\lambda \\ \mu_\beta t + \pi F_\beta \\ 0 \end{pmatrix}$$

Where the parallax factors are defined utilizing the classic formulation by Green (1985):

$$\begin{aligned} F_\lambda &= -\sin(\lambda_b - \lambda_\odot) \\ F_\beta &= -\sin\beta_b \cos(\lambda_b - \lambda_\odot) \end{aligned}$$

and $\lambda_\odot$ is the sun's longitude at the given time t. The term describing the Keplerian motion of the j-th planet in the tangent plane is:

$$\mathbf{x}_{\rm K,j} = \begin{pmatrix} {\rm x}_{\rm K,j} \\ {\rm y}_{\rm K,j} \\ 0 \end{pmatrix} = \begin{pmatrix} \varrho_{\rm j} \cos\vartheta_{\rm j} \\ \varrho_{\rm j} \sin\vartheta_{\rm j} \\ 0 \end{pmatrix},$$

where $\varrho_{\rm j}$ is the separation and $\vartheta_{\rm j}$ the position angle. The two coordinates ${\rm x}_{\rm K,j}$ and ${\rm y}_{\rm K,j}$ are functions of the 7 orbital elements:

$$\begin{aligned} {\rm x}_{\rm K,j} &= a_{\rm j}(1 - e_{\rm j}\cos E_{\rm j})(\cos(\nu_{\rm j} + \omega_{\rm j})\cos\Omega_{\rm j} - \sin(\nu_{\rm j} + \omega_{\rm j})\sin\Omega_{\rm j}\cos i_{\rm j}) \quad (3) \\ {\rm y}_{\rm K,j} &= a_{\rm j}(1 - e_{\rm j}\cos E_{\rm j})(\cos(\nu_{\rm j} + \omega_{\rm j})\sin\Omega_{\rm j} + \sin(\nu_{\rm j} + \omega_{\rm j})\cos\Omega_{\rm j}\cos i_{\rm j}), \quad (4) \end{aligned}$$

where $i_{\rm j}$ is the inclination of the orbital plane, $\omega_{\rm j}$ is the longitude of the pericenter, $\Omega_{\rm j}$ is the position angle of the line of nodes, $e_{\rm j}$ is the eccentricity, $a_{\rm j}$ is the apparent semi-major axis of the star's orbit around the system barycenter, i.e. the *astrometric signature*. For what concerns E_j, the eccentric anomaly, is the solution to Kepler's Equation:

$$E_j - e_j \sin E_j = M_j, \tag{5}$$

with the mean anomaly M_j, expressed in terms of the orbital period P_j and the epoch of the pericenter passage τ_j:

$$M_j = \frac{2\pi}{P_j}(t - \tau_j) \tag{6}$$

Finally, the true anomaly ν_j is a function of the eccentricity and the eccentric anomaly:

$$\nu_j = 2 \arctan \left\{ \left(\frac{1 + e_j}{1 - e_j} \right)^{1/2} \tan E_j/2 \right\} \tag{7}$$

We then rotate on the ecliptic reference frame by means of the transformation matrix:

$$\mathrm{R}(\lambda_b, \beta_b) = \begin{pmatrix} -\sin\lambda_b & -\sin\beta_b\cos\lambda_b & \cos\beta_b\cos\lambda_b \\ \cos\lambda_b & -\sin\beta_b\sin\lambda_b & \cos\beta_b\sin\lambda_b \\ 0 & \cos\beta_b & \sin\beta_b \end{pmatrix}$$

The other two vectors in Eq. 2 are thus defined as:

$$\begin{aligned} \mathbf{x}_{\mathrm{ecl}}^{\mathrm{K,j}} &= \mathrm{R}(\lambda_b, \beta_b) \cdot \mathbf{x}_{\mathrm{K,j}} \\ &= \begin{pmatrix} -\sin\lambda_b \varrho_j \cos\vartheta_j - \sin\beta_b \cos\lambda_b \varrho_j \sin\vartheta_j \\ \cos\lambda_b \varrho_j \cos\vartheta_j - \sin\beta_b \sin\lambda_b \varrho_j \sin\vartheta_j \\ \cos\beta_b \varrho_j \sin\vartheta_j \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{x}_{\mathrm{ecl}}^{\pi,\mu} &= \mathrm{R}(\lambda_b, \beta_b) \cdot \mathbf{x}_{\pi,\mu} \\ &= \begin{pmatrix} -\sin\lambda_b\{\mu_\lambda t + \pi F_\lambda\} - \sin\beta_b\cos\lambda_b\{\mu_\beta t + \pi F_\beta\} \\ \cos\lambda_b\{\mu_\lambda t + \pi F_\lambda\} - \sin\beta_b\sin\lambda_b\{\mu_\beta t + \pi F_\beta\} \\ \cos\beta_b\{\mu_\beta t + \pi F_\beta\} \end{pmatrix} \end{aligned}$$

This allows us to write Eq. 2 in the form:

$$\begin{pmatrix} x_{\mathrm{ecl}} \\ \\ y_{\mathrm{ecl}} \\ \\ z_{\mathrm{ecl}} \end{pmatrix} = \begin{pmatrix} \cos\beta_b\cos\lambda_b - \sin\lambda_b\varrho_j\cos\vartheta_j - \sin\beta_b\cos\lambda_b\sum_{j=1}^{n_p}\varrho_j\sin\vartheta_j \\ -\sin\lambda_b\{\mu_\lambda t + \pi F_\lambda\} - \sin\beta_b\cos\lambda_b\{\mu_\beta t + \pi F_\beta\} \\ \cos\beta_b\sin\lambda_b + \cos\lambda_b\sum_{j=1}^{n_p}\varrho_j\cos\vartheta_j - \sin\beta_b\sin\lambda_b\sum_{j=1}^{n_p}\varrho_j\sin\vartheta_j \\ +\cos\lambda_b\{\mu_\lambda t + \pi F_\lambda\} - \sin\beta_b\sin\lambda_b\{\mu_\beta t + \pi F_\beta\} \\ sin\beta_b + \cos\beta_b\sum_{j=1}^{n_p}\varrho_j\sin\vartheta_j + \cos\beta_b\{\mu_\beta t + \pi F_\beta\} \end{pmatrix}$$

Finally, a rotation to the local reference frame defined by the Instantaneous Great Circles is made by means of the transformation matrix (see for example the 3-volume ESA publication ESA-SP 1111, Perryman et al. 1989):

$$\mathbf{x}_{\mathrm{IGC}} = \mathrm{R}(\lambda_p, \beta_p) \cdot \mathbf{x}_{\mathrm{ecl}}, \tag{8}$$

where:

$$\mathrm{R}(\lambda_p, \beta_p) = \begin{pmatrix} -\sin\lambda_p & \cos\lambda_p & 0 \\ -\sin\beta_p\cos\lambda_p & -\sin\beta_p\sin\lambda_p & \cos\beta_p \\ \cos\beta_p\cos\lambda_p & \cos\beta_p\sin\lambda_p & \sin\beta_p \end{pmatrix}$$

and λ_p, β_p are the coordinates of the pole of the IGC at any given time. The resulting vector can be expressed in terms of the two angular coordinates ψ and η:

$$\mathbf{x}_{\rm IGC} = \begin{pmatrix} x_{\rm IGC} \\ y_{\rm IGC} \\ z_{\rm IGC} \end{pmatrix} = \begin{pmatrix} \cos\psi\cos\eta \\ \cos\eta\sin\psi \\ \sin\eta \end{pmatrix}$$

By now expanding in Taylor Series to first order the IGC cartesian position vector of each target, it is possible to derive a set of linearized equations of condition expressing only the observed abscissa ψ as a function of all astrometric parameters and orbital elements. We formally have:

$$\delta\mathbf{x}_{\rm IGC} = \sum_{m=1}^{n} \frac{\partial\mathbf{x}_{\rm IGC}}{\partial a_m} \mathrm{d}a_m \tag{9}$$

The n unknowns a_m represent positions, proper motions, parallax, and the $7 \star n_p$ orbital elements (if the star is not single). Now consider that:

$$\begin{aligned}
\delta\mathbf{x}_{\rm IGC} &= \delta(x_{\rm IGC}, y_{\rm IGC}, z_{\rm IGC}) = (\delta(\cos\psi\cos\eta), \delta(\sin\psi\cos\eta), \delta\sin\eta) \\
&= (-\sin\psi\cos\eta\mathrm{d}\psi - \sin\eta\cos\psi\mathrm{d}\eta, \cos\psi\cos\eta\mathrm{d}\psi - \sin\eta\sin\psi\mathrm{d}\eta, \cos\eta\mathrm{d}\eta) \\
&= (-\sin\psi\cos\eta\mathrm{d}\psi, \cos\psi\cos\eta\mathrm{d}\psi, 0) \\
&\quad +(-\sin\eta\cos\psi\mathrm{d}\eta, -\sin\eta\sin\psi\mathrm{d}\eta, \cos\eta\mathrm{d}\eta) \\
&= \cos\eta(-\sin\psi, \cos\psi, 0)\mathrm{d}\psi + (-\sin\eta\cos\psi, -\sin\eta\sin\psi, \cos\eta)\mathrm{d}\eta \\
&= \cos\eta\mathrm{d}\psi\mathbf{e}_\psi + \mathrm{d}\eta\mathbf{e}_\eta
\end{aligned}$$

where $\mathbf{e}_\eta$ and $\mathbf{e}_\psi$ constitute the pair of orthogonal unit vectors in the directions parallel to ψ and η, as defined in the tangent plane. We then have:

$$\cos\eta\mathrm{d}\psi\mathbf{e}_\psi + \mathrm{d}\eta\mathbf{e}_\eta = \sum_{m=1}^{n} \frac{\partial\mathbf{x}_{\rm IGC}}{\partial a_m} \mathrm{d}a_m \tag{10}$$

By taking the scalar product with $\mathbf{e}_\psi$, we obtain the following scalar expression:

$$\cos\eta\mathrm{d}\psi = (-\sin\psi) \sum_{m=1}^{n} \frac{\partial x_{\rm IGC}}{\partial a_m} \mathrm{d}a_m + (\cos\psi) \sum_{m=1}^{n} \frac{\partial y_{\rm IGC}}{\partial a_m} \mathrm{d}a_m \tag{11}$$

If we now define:

$$c_{a_m} = (-\sin\psi) \frac{\partial x_{\rm IGC}}{\partial a_m} + (\cos\psi) \frac{\partial y_{\rm IGC}}{\partial a_m}, \tag{12}$$

then the linearized condition equation takes the form:

$$\cos\eta\mathrm{d}\psi = \sum_{m=1}^{n} c_{a_m} \mathrm{d}a_m = F(\lambda, \beta, \mu_\lambda, \mu_\beta, \pi, a_{\rm j}, P_{\rm j}, \tau_{\rm j}, \omega_{\rm j}, \Omega_{\rm j}, e_{\rm j}, i_{\rm j}), \;\; j = 1, \ldots, n_P \tag{13}$$

For each given target, there will be as many equations of this form as the number of observation epochs. The quantity $\mathrm{d}\psi = \psi_{\rm obs} - \psi_{\rm cat}$ is defined as the difference between the observed and catalog abscissa.