The use of Docobo's analytic method for calculating visual double star orbits

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Introduction

Docobo's analytic method for calculating binary star orbits has proved be a very useful, versatile, and friendly application. Different astronomers have used the method to calculate more than 300 orbits during the last decades. It is based on an application from the interval $(0,2\pi)$ or $(0,\infty)$ into the set of periodic Keplerian orbits whose corresponding apparent orbits pass through the three base points, selecting the corresponding orbit in each case checking the residuals obtained with all the observations available or using other criteria. From a mathematical perspective, the algorithm developed can be consider to be the natural way to analytically determine the seven orbital elements in a simple way. As a result, the Thiele-Innes-Van den Bos and Cid methods are particular cases of it. In this communication, I present a summary of the method and its more important applications.
The first solution to the problem of calculating the elliptic orbit of a visual double star given the necessary and sufficient data was that of Thiele-Innes-Van den Bos [8, 9, 10] which employs three complete observations of the form $(\theta,\rho;t)$ together with the double areal constant, $\mathscr{C}$, which must be obtained from additional data. This method has served as the basis for a great deal of research, particularly that conducted by S. Arend and J. Dommanget at the Royal Observatory of Belgium.
In 1958 Cid [1, 2] developed a direct method involving only observational data, namely three complete $(\theta_i,\rho_i;t_i)_{i=1,2,3}$ observations and an incomplete observation of the form $(\theta_4;t)$. It is clear that progressively changing the value of $\mathscr{C}$ in the Thiele-Innes-Van den Bos method [7] or that of the fourth angle in Cid's method, produces a series of different orbits that all pass through the three given points. In other words, three observations define a (possibly empty) set of Keplerian orbits whose corresponding apparent orbits pass through those points (we will call this set, $\mathscr{E}$). This is the idea of the method designed by J. A. Docobo [3] who established a simple algorithm that, using the base points $(\theta_i,\rho_i;t_i)_{i=1,2,3}$, permits the establishment of a mapping from the interval $(0,2\pi)$ to the set, $\mathscr{E}$.

$$\begin{aligned}(0,2\pi) &\longrightarrow \mathscr{E} \\ V &\longmapsto (P,T,e,a",i,\Omega,\omega)\end{aligned} \tag{1}$$

If these base points belong to different revolutions, it is necessary to substitute the interval $(0,2\pi)$ with $(0,\infty)$.
Obviously, the three base points must be observations with great weight or they are "virtual" points belonging to areas with a maximum degree of observational evidence in their favor.

1. Notation

$P,T,e,a",i,\Omega,\omega$ have their usual meaning for orbits.
$n=\frac{2\pi}{P}$: mean annual motion
t_i : time at which the secondary star occupies its ith position / $t_{ij}=t_j-t_i$
E_i : eccentric anomaly at time t_i / $E_{ij}=E_j-E_i$
$V=E_{13},\ U=E_{23}$
$F(x)=x-\sin x$
θ_i,ρ_i : coordinates of the secondary star at time t_i / $\Delta_{ij}=\rho_i\rho_j\sin(\theta_j-\theta_i)$
$\mathscr{C}$ = the double areal constant of the apparent orbit

$R = \frac{\Delta_{12}}{\Delta_{13}}$, $S = \frac{\Delta_{12}}{\Delta_{23}}$
$\mathscr{E}$: set of periodic Keplerian orbits whose corresponding apparent orbits pass through the three points defined by $(\theta_i, \rho_i; t_i)$, $1 \leq i \leq 3$

2. The calculation process

If, for the three pairs $(t_1,t_2),(t_2,t_3)$ and (t_3,t_1), the constant $\mathscr{C}$ is eliminated from among the Thiele equations

$$\begin{aligned} t_2 - t_1 - \frac{\Delta_{12}}{\mathscr{C}} &= \frac{1}{n}F(V-U) \\ t_3 - t_2 - \frac{\Delta_{23}}{\mathscr{C}} &= \frac{1}{n}F(U) \\ t_3 - t_1 - \frac{\Delta_{13}}{\mathscr{C}} &= \frac{1}{n}F(V) \end{aligned} \tag{2}$$

the result is

$$\begin{aligned} n(t_{12} - Rt_{13}) &= F(V-U) - RF(V) \\ n(t_{12} - St_{23}) &= F(V-U) - SF(U) \end{aligned} \tag{3}$$

Futher eliminating the mean annual motion leaves the equation

$$\Sigma - \varepsilon \sin \Sigma = \Psi \tag{4}$$

with

$$\begin{aligned} \varepsilon \sin W &= \frac{\sin V}{1-N} \\ \varepsilon \cos W &= \frac{\cos V - N}{1-N} \\ \Sigma &= U - W \\ \Psi &= \frac{t_{23}}{t_{13}}F(V) + \varepsilon \sin W - W \end{aligned} \tag{5}$$

As we have seen, the independent variable, V, is the difference between the eccentric anomalies that correspond to points 3 and 1; as such, U corresponds to points 3 and 2. For each value of V, the correspondent of U is obtained by means of the resolution of an equation (4) that is an analog to Kepler's equation. The sequence used in order to obtain the elements is the following:

$$
\begin{array}{c}
V \\
\downarrow \\
n = \dfrac{2\pi}{P} = \dfrac{F(V-U) - RF(V)}{t_{12} - R - t_{13}} \leftarrow U \rightarrow \mathscr{C} \\
\downarrow \left\{ \begin{array}{ll} e^2 = 1 + \frac{S(1+cos\,U)+1-cos(V-U)-R(1-cos\,V)}{Q} & (*) \\ sin\,E_2 = \varphi_1(\Delta_{ij}, e, U, V) & \\ cos\,E_2 = \varphi_2(\Delta_{ij}, e, U, V) & \end{array} \right\} \qquad (6) \\
e, E_2 \\
\swarrow \quad \searrow \\
\begin{array}{r} E_1, E_3 \\ (Kepler\ eq.) \downarrow \\ T \end{array} \quad \begin{array}{l} a = \varphi_3(\rho_i, \rho_j, \theta_i\theta_j, e, E_i, E_j) \\ i = \varphi_4(\rho_i, \rho_j, \theta_i\theta_j, e, E_i, E_j) \\ \Omega = \varphi_5(\rho_i, \rho_j, \theta_i\theta_j, e, E_i, E_j) \\ \omega = \varphi_6(\rho_i, \rho_j, \theta_i\theta_j, e, E_i, E_j) \end{array}
\end{array}
$$

Once we have orbital elements that correspond to each value of V, it is necessary to check the residuals obtained with the rest of the available observations, taking into account the corresponding weight.
In this way, we can calculate the root mean square (RMS) and the mean absolute (MA) in θ and in ρ. That orbit with a lesser RMS or lesser MA will be selected in each case.

3. Other ways to select the orbit

This method also permits other alternatives to select the final orbit. In effect, previously introducing among the data the spectral types and magnitudes and the Hipparcos parallax, we can obtain the followings outputs:
a) **Dynamic parallax**: this allows us to check the dynamic parallax obtained for each orbit with the Hipparcos parallax.
b) **Total system mass**: using the Hipparcos parallax, we can calculate the dynamic mass for the orbit and compare it with the typical value of the mass for corresponding spectral types, or even with the calculated mass using empirical calibrations.
c) **The double areal constant of the apparent orbit, $\mathscr{C}$**: by means of the expression $\mathscr{C} = \frac{\Delta_{13}}{t_3 - t_1 - \frac{1}{n}F(V)}$, we can determine $\mathscr{C}$ and for each orbit and check it with the previously calculated value as done with the Thiele-Innes-Van den Bos method.
d) **The position angle in a fourth epoch,** θ_4: And so, we are in the same conditions as with Cid's method.
The outputs c) and d) show us that the methods of Thiele-Innes – Van den Bos, and Cid can be considered to be particular cases of Docobo's method.

4. Previous study of the existence of periodic solutions

It is interesting, although not necessary, to conduct a prior study of the existence of periodic orbits whose apparent orbits pass through the 3 base points. In this sense, the behavior of the function (*) (Eq. 6) that gives us the eccentricity is the key.
The relative positions of the three points determine four situations characterized by the signs of $R = \frac{\Delta_{12}}{\Delta_{13}}$ and $S = \frac{\Delta_{12}}{\Delta_{23}}$:

Case 1: $R < 0,\ S > 0$. This case, in which the principal star is an interior point of the triangle formed by the three positions, is the only one that guarantees the existence of elliptic orbits for whatever value of V and whatever values of t_i.
Case 2: $R > 0,\ S > 0$. In this situation, the periodic orbits, it they exist, are at the beginning of the interval.
Case 3: $R < 0,\ S < 0$. Now, the elliptic orbits appear at the end of the interval $(0, 2\pi)$.
Case 4: $R > 0,\ S < 0$. In this fourth case, if elliptic orbits exist, they are in the middle of the interval.
In the original publication [3], the necessary and sufficient conditions are given for the existence of solutions in each one of these cases. In the particular cases in which there are opposite points among the base points, the existence of periodic orbits is guaranteed.

5. An application example

Gliese 22 is a hierarchical triple star system, perhaps even quadruple now that we postulated the existence of a fourth object of low mass within it [6], formed by 3 red dwarfs whose individual masses can be obtained by empirical calibrations, yielding: component, Aa ($0.377 \pm 0.030 M_\odot$); component, Ab (0.138 ± 0.007), and component, B (0.177 ± 0.014). Total mass, 0.692 ± 0.034. The orbit of Ab with respect to Aa is almost definite and the orbit of B with respect to the center of mass of Aa and Ab only has an arc of 80°. Even so it is possible to obtain a more probable orbit, selecting among all that fit the observations, that which yields a dynamic mass that is similar to that which was empirically predetermined. Next, the process of approximation to the selected solution is presented.

V (°)	P (yr)	T	e	a (")	i (°)	Ω (°)	ω (°)	rms_θ	rms_ρ	Total mass $M_\odot$	
First approximation											
90	336.9	1928.8	0.046	4.175	52.2	162.0	292.1	0.96	0.11	0.60	
110	198.0	1870.7	0.411	3.181	46.9	2.1	316.4	1.32	0.15	0.77	
130	130.7	1902.1	0.914	5.669	71.2	31.9	278.7	1.32	0.14	10.05	
Second approximation											
103	234.9	1854.77	0.249	3.387	47.8	172.2	150.0	0.96	0.11	0.66	
104	229.0	1857.17	0.271	3.347	47.5	173.4	148.1	0.97	0.11	0.67	
105	**223.4**	**1859.54**	**0.294**	**3.310**	**47.3**	**174.7**	**146.2**	**0.97**	**0.11**	**0.69**	←
106	217.9	1861.86	0.317	3.277	47.1	176.0	144.3	0.97	0.11	0.70	
107	212.7	1864.15	0.340	3.247	47.0	177.4	142.4	0.97	0.10	0.71	

Conclusion

Docobo's analytic method for the calculus of visual double star orbits has been applied successfully by various astronomers with the result of obtaining 300 orbits in the last decade. The algorithm, which is easy to program, constitutes a friendly calculus process not only for calculating orbits but also for the previous study of periodic solutions. Here, the advantages of Docobo's method are highlighted.

1. It is useful for determining orbits even when we have observations that belong to different revolutions.
2. It can be utilized for the calculation of orbits with an inclination of 90°.
3. It permits the selection of the desired orbit, taking into account the total mass as well as the parallax.

4. Both the method of Cid as well as that of Thiele-Innes-Van den Bos can be considered to be particular cases of it.

5. It permits the calculation of orbits with short arcs.

6. It is useful for determining the exterior orbit of triple systems once the interior orbit is known.

7. We have extended the method to include radial velocities among the data making the calculus of orbits possible using mixed data [5].

8. Different applications of the method can be seen in many papers published in international journals.

9. The program of Docobo's method is currently available in the computer languages, FORTRAN and MATLAB [4].

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